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H_2 -reducible complex Hadamard matrices of order 6

Bengt R. Karlsson

Uppsala University, Department of Physics and Astronomy, Box 516, SE-751 20 Uppsala, Sweden

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ABSTRACT

Complex Hadamard matrices H of order 6 are characterized in a novel manner, according to the presence/absence of order 2 Hadamard submatrices. It is shown that if there exists one such submatrix, H is equivalent to a Hadamard matrix where all the nine submatrices are Hadamard. The ensuing subset of H_2 -reducible complex Hadamard matrices is more general than might be thought, and, significantly, includes all the up till now described (one- and two-parameter) families of order 6. A known, isolated matrix, and most numerically generated matrices, fall outside the subset.

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1. Introduction

Complex Hadamard matrices (for an overview, see [1,2]) have recently become a topic of interest, in part because of the correspondence between such matrices and mutually unbiased bases, MUBs. Particular attention has been given to two unsettled problems in six dimensions, see for instance [3]. On the one hand, six is the lowest order for which a complete characterization of the complex Hadamard matrices is lacking, and, on the other hand, it is also the lowest dimension for which a full understanding of the MUBs is missing. These two problems are not necessarily (directly) related, but progress in one may have implications for the other.

There are good reasons for expecting most complex Hadamard matrices of order 6 to be elements in a four-parameter set [3,4], but up till now only one- and two-parameter subsets have been described in closed form. Recent progress includes the identification of three new two-parameter families [5,6] that, together with the two Fourier families, incorporate all previously described one-parameter families as subfamilies. These five two-parameter families are partially overlapping, indicating that they might

E-mail address: bengt.karlsson@physics.uu.se

have some unidentified common feature relevant for a more comprehensive characterization. A clue to what this feature might be was found in [6] where it was observed that the matrices of the discovered two-parameter family, $K_6^{(2)}$ in the notation of [2], could be seen as composed of nine 2×2 Hadamard submatrices.

In the present paper, it is shown that the set of Hadamard matrices having such a substructure includes not only $K_6^{(2)}$ but also all other so far described one- and two-parameter Hadamard families (disregarding families for which there only exists numerical evidence). More generally, it is shown that any complex Hadamard matrix of order 6 is equivalent to a matrix where either all or none of the nine 2×2 submatrices are Hadamard; this is the main result of the present paper. In a separate paper [7] it will be shown how the subset of H_2 -reducible matrices can be fully described in closed form as a three-parameter Hadamard family.

2. Preliminaries

The Hadamard matrices of interest here differ from the more common ones in that the elements are not restricted to 1 or -1 but can be any complex number on the unit circle.

Definition 1. A square matrix H with complex elements h_{ij} is Hadamard if $|h_{ij}| = 1$, and if

$$HH^\dagger = H^\dagger H = NE \quad (2.1)$$

Here, N is the order of H , and E is the unit matrix of order N .

The condition (2.1) will be referred to as the unitarity constraint on H , with the understanding that it is the matrix H/\sqrt{N} that is unitary. Furthermore, $HH^\dagger = NE$ implies $H^\dagger H = NE$, and vice versa.

Definition 2. Two Hadamard matrices are termed equivalent, $H_1 \sim H_2$, if they can be related through

$$H_2 = D_2 P_2 H_1 P_1 D_1 \quad (2.2)$$

where D_1 and D_2 are diagonal, unitary matrices and P_1 and P_2 are permutation matrices.

A set of equivalent Hadamard matrices can be represented by a dephased matrix, with all elements in the first row and the first column equal to 1. For order 2, all Hadamard matrices are equivalent to the dephased matrix

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.3)$$

For orders 3, 4 and 5, all inequivalent complex Hadamard matrices have been fully characterized, while for order 6 the characterization is far from complete. Currently it is based on an isolated matrix $S_6^{(0)}$, on the two-parameter (Fourier) families $F_6^{(2)}$ and $(F_6^{(2)})^T$, and on the three recently reported two-parameter families $K_6^{(2)}$, $X_6^{(2)}$ and $(X_6^{(2)})^T$ (all in the notation of [1,2]).

As a step in the search for a more comprehensive characterization, the following subset of Hadamard matrices is identified.

Definition 3. A complex Hadamard matrix of order 6 is H_2 -reducible if it is equivalent to a Hadamard matrix for which all the nine 2×2 submatrices are Hadamard.

The introduction and investigation of H_2 -reducible Hadamard matrices has turned out to be rewarding, as is detailed in the next section and in [7].

3. H_2 -reducible Hadamard matrices

H_2 -reducible Hadamard matrices are more prevalent than might be thought. The general nature of these matrices is made clear by the following theorem, which also contains the main result of the present paper.

Theorem 4. *Let H be a Hadamard matrix of order 6, with elements h_{ij} , $i, j = 1, \dots, 6$. If there exists an order 2 submatrix $\begin{pmatrix} h_{ij} & h_{ik} \\ h_{lj} & h_{lk} \end{pmatrix}$ that is Hadamard, then H is H_2 -reducible.*

As a corollary it will be seen that all currently known one- and two-parameter Hadamard families are equivalent to subsets in the set of H_2 -reducible Hadamard matrices; in contrast, the isolated matrix $S_6^{(0)}$ turns out not to be H_2 -reducible.

The proof of Theorem 4 proceeds in several steps. First recall the following properties of the elements of Hadamard matrices.

Lemma 5. *Let z_1, \dots, z_4 be four complex numbers on the unit circle. If $z_1 + z_2 + z_3 + z_4 = 0$, then for each z_i there is a z_j such that $z_i + z_j = 0$.*

The proof is immediate since the relation $z_1 + z_2 + z_3 + z_4 = 0$ corresponds to a (possibly degenerate) rhomb in the complex plane.

Lemma 6. *Let z_1 and z_2 be two complex numbers on the unit circle such that $z_2 \neq \pm z_1$. If $\operatorname{Re}(z_1 w) = \operatorname{Re}(z_2 w) = 0$ for some complex number w , then $w = 0$.*

Again, the proof is elementary.

Proposition 7. *Let H be a Hadamard matrix of order 6 with elements h_{ij} , $i, j = 1, \dots, 6$. If there exists an order 2 submatrix $\begin{pmatrix} h_{ij} & h_{ik} \\ h_{lj} & h_{lk} \end{pmatrix}$ that is Hadamard, then H is equivalent to a dephased Hadamard matrix on the form*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & z_1 & -z_1 & z_2 & -z_2 \\ 1 & z_3 & \bullet & \bullet & \bullet & \bullet \\ 1 & -z_3 & \bullet & \bullet & \bullet & \bullet \\ 1 & z_4 & \bullet & \bullet & \bullet & \bullet \\ 1 & -z_4 & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \quad (3.1)$$

Proof. Through permutation of rows and columns, the submatrix $\begin{pmatrix} h_{ij} & h_{ik} \\ h_{lj} & h_{lk} \end{pmatrix}$ can be brought to the upper left corner of H . A subsequent dephasing turns it into $F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and an overall dephasing results in a matrix on the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & u_1 & u_2 & u_3 & u_4 \\ 1 & w_1 & \bullet & \bullet & \bullet & \bullet \\ 1 & w_2 & \bullet & \bullet & \bullet & \bullet \\ 1 & w_3 & \bullet & \bullet & \bullet & \bullet \\ 1 & w_4 & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

where all u_i and w_i are on the unit circle. The unitarity constraint now requires that $u_1 + u_2 + u_3 + u_4 = 0$, and that $w_1 + w_2 + w_3 + w_4 = 0$. These relations can only be satisfied if to each u_i there is a

$u_k = -u_i$, and similarly for w_i (Lemma 5). A final permutation of rows and of columns, and a renaming of the entries, leaves the matrix on the standard form (3.1). \square

At this point it is convenient to introduce the four Hadamard matrices

$$\begin{aligned} Z_1 &= \begin{pmatrix} 1 & 1 \\ z_1 & -z_1 \end{pmatrix} & Z_2 &= \begin{pmatrix} 1 & 1 \\ z_2 & -z_2 \end{pmatrix} \\ Z_3 &= \begin{pmatrix} 1 & z_3 \\ 1 & -z_3 \end{pmatrix} & Z_4 &= \begin{pmatrix} 1 & z_4 \\ 1 & -z_4 \end{pmatrix} \end{aligned} \quad (3.2)$$

and write the matrix (3.1) on block form

$$H = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & a & b \\ Z_4 & c & d \end{pmatrix} \quad (3.3)$$

The remaining task is to show that any Hadamard matrix of the type specified in Theorem 4 is equivalent to (or equals) a matrix on the form (3.3) where also the four 2×2 submatrices a, b, c , and d are Hadamard.

Proposition 8. *If a Hadamard matrix has the form (3.3), and one of the matrices a, b, c and d is Hadamard, then the other three are also Hadamard.*

Proof. The unitarity constraints (2.1) imply, among other relations, that

$$aa^\dagger + bb^\dagger = 4e \quad (3.4)$$

$$cc^\dagger + dd^\dagger = 4e \quad (3.5)$$

$$a^\dagger a + c^\dagger c = 4e \quad (3.6)$$

$$b^\dagger b + d^\dagger d = 4e \quad (3.7)$$

Let a be Hadamard. Then the relations (3.4) and (3.6) reduce to $bb^\dagger = 2e$ and $c^\dagger c = 2e$, i.e. b and c are also Hadamard. It now follows from (3.5) that $dd^\dagger = 2e$, i.e. also d is Hadamard. Similar arguments apply if b, c or d is chosen as the initially Hadamard matrix. \square

For completeness, the following result from [6] is included here. This result initiated the present investigation.

Theorem 9. *If a Hadamard matrix has the form (3.3), and $Z_1 = Z_2$ and $Z_3 = Z_4$, then H is equivalent to a Hadamard matrix on the same form where a, b, c and d are Hadamard, with $a = d$ and $b = c$.*

Proof. The unitarity constraints (2.1) give rise to four linear relations between a, b, c and d ,

$$a + b = a + c = b + d = c + d = -Z \quad (3.8)$$

where $Z = Z_3 F_2 Z_1 / 2$. These relations imply that $d = a$ and $c = b$, and that the remaining unitarity constraints can be simplified, to read

$$(a - b)^\dagger(a - b) = (a - b)(a - b)^\dagger = 6e. \quad (3.9)$$

The matrix Z has the property that $Z^\dagger Z = Z Z^\dagger = 2e$, and the matrix elements satisfy the relations $Z_{21} = z_1 z_3 \bar{z}_{12}$, $Z_{22} = -z_1 z_3 \bar{z}_{11}$ and $|Z_{ij}|^2 \leq 2$.

Since the modulus of each element of a and b is one, the relation $a + b = -Z$ can be solved element by element,

$$a_{ij} = -Z_{ij} \left(\frac{1}{2} + i\sigma_{ij} \sqrt{\frac{1}{|Z_{ij}|^2} - \frac{1}{4}} \right)$$

$$b_{ij} = -Z_{ij} \left(\frac{1}{2} - i\sigma_{ij} \sqrt{\frac{1}{|Z_{ij}|^2} - \frac{1}{4}} \right)$$

where $\sigma_{ij} = \pm 1$. These expressions apply also if for some values of z_1 and z_3 , and some pair of indices i and j , $|Z_{ij}| \rightarrow 0$ with $Z_{ij}/|Z_{ij}| = \exp(i \arg Z_{ij})$. Note that the elements of Z are related, so that if for instance $|Z_{11}| \rightarrow 0$, then $|Z_{22}| = |Z_{11}| \rightarrow 0$, $|Z_{12}| = |Z_{21}| \rightarrow \sqrt{2}$, with $Z_{22}/|Z_{22}| = -\bar{Z}_{11}Z_{21}/(|Z_{11}||\bar{Z}_{12}|)$ (from the condition $ZZ^\dagger = 2e$).

The relations (3.9) simply impose further constraints on the sign factors σ_{ij} ,

$$\sigma_{11}\sigma_{21} = \sigma_{12}\sigma_{22} \quad (3.10)$$

Through permutation of the rows and/or the columns of H , it can be verified that all sign combinations compatible with (3.10) correspond to equivalent matrices. For some of these matrices, the sign factors are related through $\sigma_{11} + \sigma_{22} = \sigma_{12} + \sigma_{21} = 0$, and in these cases $a^\dagger a = b^\dagger b = 2e$. Therefore, H is equivalent to a matrix for which all the 2×2 submatrices are Hadamard, as was to be shown. \square

Theorem 4 can now be proven.

Proof. In (3.3), let $a = \frac{1}{2}Z_3AZ_1$, $b = \frac{1}{2}Z_3BZ_2$, $c = \frac{1}{2}Z_4CZ_1$ and $d = \frac{1}{2}Z_4DZ_2$. The unitarity constraints (2.1) on H give rise to four linear relations between A , B , C and D ,

$$\begin{cases} A & + & B & = & -F_2 \\ C & + & D & = & -F_2 \\ A & + & C & = & -F_2 \\ B & + & D & = & -F_2 \end{cases} \quad (3.11)$$

and these relations imply that $D = A$ and $C = B$. As a result,

$$\begin{aligned} a &= \begin{pmatrix} a_{11}(z_3, z_1) & a_{11}(z_3, -z_1) \\ a_{11}(-z_3, z_1) & a_{11}(-z_3, -z_1) \end{pmatrix} \\ b &= \begin{pmatrix} b_{11}(z_3, z_2) & b_{11}(z_3, -z_2) \\ b_{11}(-z_3, z_2) & b_{11}(-z_3, -z_2) \end{pmatrix} \\ c &= \begin{pmatrix} b_{11}(z_4, z_1) & b_{11}(z_4, -z_1) \\ b_{11}(-z_4, z_1) & b_{11}(-z_4, -z_1) \end{pmatrix} \\ d &= \begin{pmatrix} a_{11}(z_4, z_2) & a_{11}(z_4, -z_2) \\ a_{11}(-z_4, z_2) & a_{11}(-z_4, -z_2) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11}(z_3, z_1) &= (A_{11} + z_1A_{12} + z_3A_{21} + z_1z_3A_{22})/2 \\ b_{11}(z_3, z_2) &= (B_{11} + z_2B_{12} + z_3B_{21} + z_2z_3B_{22})/2 \end{aligned}$$

If it can be shown that A satisfies the unitarity constraint $A^\dagger A = 2e$, then $a^\dagger a = 2e$, a is Hadamard, and a reference to Propositions 7 and 8 completes the proof.

The elements of A are constrained by the condition that all elements of a , b , c and d are on the unit circle, and this condition is sufficient to ensure that $A^\dagger A = 2e$. Indeed, from the conditions $|a_{ij}| = 1$ one finds

$$|A_{11}|^2 + |A_{12}|^2 + |A_{21}|^2 + |A_{22}|^2 = 4 \quad (3.12)$$

$$\operatorname{Re} \left(z_3 (A_{21}\bar{A}_{11} + A_{22}\bar{A}_{12}) \right) = 0 \quad (3.13)$$

$$\operatorname{Re} \left(z_1 (A_{12}\bar{A}_{11} + A_{22}\bar{A}_{21}) \right) = 0 \quad (3.14)$$

$$\operatorname{Re} \left(z_1z_3A_{22}\bar{A}_{11} + \frac{z_1}{z_3}A_{12}\bar{A}_{21} \right) = 0 \quad (3.15)$$

The conditions on the elements of d give rise to a similar set of equations, with $z_1 \rightarrow z_2$ and $z_3 \rightarrow z_4$. From the elements of b and c there are two more sets, which are obtained from (3.12)–(3.15) by taking $A \rightarrow B$, and $z_1 \rightarrow z_2$ (for b), or $z_3 \rightarrow z_4$ (for c). The last two sets can be converted into conditions on the elements of A by means of the relation $B = -F_2 - A$. The resulting set of equations can be simplified using the relations (3.12)–(3.15), and read

$$\operatorname{Re}(A_{11} + A_{12} + A_{21} - A_{22}) = -2 \quad (3.16)$$

$$\operatorname{Re}\left(z_3 \left(\bar{A}_{11} + A_{22} + A_{21} - \bar{A}_{12}\right)\right) = 0 \quad (3.17)$$

$$\operatorname{Re}\left(z_2 \left(\bar{A}_{11} + A_{22} + A_{12} - \bar{A}_{21}\right)\right) = 0 \quad (3.18)$$

$$\operatorname{Re}\left(z_2 z_3 (A_{22} - 1) (\bar{A}_{11} + 1) + \frac{z_2}{z_3} (A_{12} + 1) (\bar{A}_{21} + 1)\right) = 0 \quad (3.19)$$

from b , and there is a similar set, with $z_2 \rightarrow z_1$ and $z_3 \rightarrow z_4$, from c . Several cases need to be distinguished.

Case 1. $z_1 \neq \pm z_2$ and $z_3 \neq \pm z_4$.

From (3.13) and the corresponding equation with $z_3 \rightarrow z_4$ it follows from Lemma 6 that

$$A_{21} \bar{A}_{11} + A_{22} \bar{A}_{12} = 0.$$

Similarly, from (3.18) and the corresponding equation with $z_2 \rightarrow z_1$ it follows that

$$\bar{A}_{11} - \bar{A}_{21} = -(A_{12} + A_{22})$$

These two relations can be combined to give

$$|A_{11}|^2 + |A_{21}|^2 = |A_{12}|^2 + |A_{22}|^2 = 2 \quad (3.20)$$

where the last equality follows from (3.12). Finally, from (3.14) and the corresponding equation with $z_1 \rightarrow z_2$ it follows that

$$\bar{A}_{11} A_{12} + \bar{A}_{21} A_{22} = 0 \quad (3.21)$$

The two relations (3.20) and (3.21) imply that the matrix A satisfies the unitarity constraint $A^\dagger A = 2e$, as was to be shown.

Case 2. $z_1 \neq \pm z_2$ but $z_3 = \pm z_4$.

From (3.14), (3.15), (3.18) and (3.19), and the corresponding relations where $z_1 \leftrightarrow z_2$, it follows that (since $z_3 = \pm z_4$ and by Lemma 6)

$$A_{12} \bar{A}_{11} + A_{22} \bar{A}_{21} = 0 \quad (3.22)$$

$$z_3 A_{22} \bar{A}_{11} + \frac{1}{z_3} A_{12} \bar{A}_{21} = 0 \quad (3.23)$$

$$\bar{A}_{11} + A_{22} + A_{12} - \bar{A}_{21} = 0 \quad (3.24)$$

$$z_3 (A_{22} - 1) (\bar{A}_{11} + 1) + \frac{1}{z_3} (A_{12} + 1) (\bar{A}_{21} + 1) = 0 \quad (3.25)$$

Combining (3.22) and (3.24), and (3.23) and (3.25), one finds the conditions

$$(\bar{A}_{11} - \bar{A}_{21}) (A_{12} - \bar{A}_{21}) = 0 \quad (3.26)$$

$$z_3 (A_{22} - \bar{A}_{11} - 1) + \frac{1}{z_3} (A_{12} + \bar{A}_{21} + 1) = 0 \quad (3.27)$$

In view of (3.26), either $A_{21} = \bar{A}_{12}$ or $A_{21} = A_{11}$.

Subcase 2.1. Let $A_{21} = \bar{A}_{12}$. By (3.24), $A_{22} = -\bar{A}_{11}$, and hence, by (3.12), $|A_{11}|^2 + |A_{21}|^2 = |A_{12}|^2 + |A_{22}|^2 = 2$. This relation, together with (3.22), implies that $A^\dagger A = 2e$.

Subcase 2.2. Let instead $A_{21} = A_{11}$. Then, by (3.24), $A_{12} = -A_{22}$, and hence, by (3.13), (3.23) and (3.27),

$$\begin{aligned}(|A_{11}|^2 - |A_{22}|^2) \operatorname{Re}(z_3) &= 0 \\ A_{22} \bar{A}_{11} \operatorname{Im}(z_3) &= 0 \\ (1 + \bar{A}_{11} - A_{22}) \operatorname{Im}(z_3) &= 0\end{aligned}$$

If here $\operatorname{Re}(z_3) \neq 0$, then $|A_{11}|^2 = |A_{22}|^2 = |A_{12}|^2 = |A_{21}|^2 = 1$ and again $A^\dagger A = 2e$ (with the additional condition that $\operatorname{Im}(z_3) = 0$, i.e. $z_3^2 = z_4^2 = 1$).

If instead $\operatorname{Re}(z_3) = 0$, so that $\operatorname{Im}(z_3) \neq 0$, then either $A_{11} = A_{21} = 0$ with $A_{22} = -A_{12} = 1$, or $A_{11} = A_{21} = -1$ with $A_{22} = -A_{12} = 0$. Neither of these conditions is compatible with the condition (3.12), expressing that there exists no Hadamard matrix on the form (3.3) such that $A_{21} = A_{11}$ and $\operatorname{Re}(z_3) = \operatorname{Re}(z_4) = 0$.

Summarizing Case 2, for H to be Hadamard, either $A_{12} = \bar{A}_{21}$ and $A_{22} = -\bar{A}_{22}$, or else $A_{11} = A_{21}$ and $A_{12} = -A_{22}$, with the additional condition $z_3^2 = z_4^2 = 1$. In either case, A satisfies the unitarity constraint $A^\dagger A = 2e$, as was to be shown.

Case 3. $z_1 = \pm z_2$ but $z_3 \neq \pm z_4$.

The arguments for this case mirror those of Case 2.

Case 4. $z_1 = \pm z_2$ and $z_3 = \pm z_4$.

This case is covered by Theorem 9.

In all cases, the matrix A therefore satisfies the unitarity constraint $A^\dagger A = 2e$, and with this result, the proof of Theorem 4 is completed. \square

From its dephased form, it is easy to see whether a Hadamard matrix is H_2 -reducible or not.

Corollary 10. *Let H be a complex Hadamard matrix of order 6. H is H_2 -reducible if, and only if, its dephased form has at least one element equal to -1 .*

Proof. If one element equals -1 , there is a submatrix which equals the 2×2 Hadamard matrix $F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and Theorem 4 applies. On the other hand, if a dephased Hadamard is reducible, the upper left corner 2×2 Hadamard submatrix must equal F_2 . \square

It follows from the corollary that all the currently known [2] one- and two-parameter families of order 6 are families of H_2 -reducible Hadamard matrices. On the other hand, the single, isolated matrix $S_6^{(0)}$ is not H_2 -reducible.

4. MUBs and H_2 -reducible Hadamard matrices

As was pointed out above, all known, closed form Hadamard families of order 6 are families of H_2 -reducible Hadamard matrices. A similar statement holds for the few cases where closed form MUB matrices are known. For instance, let $\{I, F_6(0, b), C(b)\}$ be the family of MUB triplets as presented in Theorem 2.4 of [10]. As is easily verified, for each b , $C(b)$ is equivalent to $F_6(0, b')$ for some b' , and $C(b)$, like $F_6(0, b)$, therefore belongs to the set of H_2 -reducible Hadamard matrices. Similarly, Zauner's construction [12], as quoted in [10], involves a family of triplets $\{I, E_1(x), E_2(x)\}$. For each x , the matrices E_1 and E_2 are both equivalent to $F_6(0, 0)$ (from Appendix B in [10]), and, as noted in [10], $E_1^\dagger E_2$ is equivalent to a member of the family $D_6^{(1)}$ (in the notation of [1,2]). Again, therefore, E_1, E_2 and $E_1^\dagger E_2$ are all in the set of H_2 -reducible Hadamard matrices.

5. Conclusion and outlook

In a separate paper [7] it is shown that an H_2 -reducible Hadamard matrix can be fully characterized in terms of a three-parameter family of complex Hadamard matrices of order 6. The overall picture

is therefore that the subset of H_2 -reducible Hadamard matrices has been completely characterized, and that in the process all previously described one- and two-parameter families reappear in a unified setting.

For the set of Hadamard matrices that are not H_2 -reducible, on the other hand, very little is known: it contains the isolated matrix $S_6^{(0)}$, and some of its members belong to one or several four-parameter families. In spite of recent efforts towards finding Hadamard families, not a single (analytically described) family has been found that extends into the set of non-reducible Hadamard matrices. The additional information that has come from numerical investigations is also very limited. As expected, numerically generated Hadamard matrices are in general not H_2 -reducible, unless specifically designed to be so. Such matrices can also be designed to trace out subfamilies in the non-reducible domain (from observations of some 10^5 matrices generated in a semi-random manner; see also [4]), but this is also as expected if indeed a four-parameter family exists. In all, however, the notion of H_2 -reducibility provides a new perspective also in the search for a characterization of the full set of complex Hadamard matrices of order 6.

The concept of H_2 -reducible Hadamard matrices has in this paper only been defined for order 6. It would seem worthwhile to generalize this concept to higher orders, by distinguishing the Hadamard matrices with a substructure of Hadamard blocks from those for which such a structure is absent. Based on the experience for order 6, the real challenge will most likely be to find and characterize the Hadamard matrices that lack such substructure. In this connection, note that for the special case of real Hadamard matrices (in dimensions 2 or $4n$), it has been conjectured [13] that every such matrix has a partition into 2×2 rank 2 submatrices, i.e. that such matrices are H_2 -reducible.

The possible relevance of the result obtained here for the understanding of mutually unbiased bases (MUBs) in six dimensions is left for further study. Extensive numerical searches [8,9] indicate that the maximal number of such bases is no greater than three, but an understanding of why this should be so is lacking. Similarly, MUBs, like the Hadamard matrices, come in families [10,5,11], but a full characterization of for instance all triplets of MUBs in six dimensions has so far not been achieved. Interestingly, all currently known (to us) Hadamard members of MUB triplets are H_2 -reducible, even those obtained through numerical searches, and if this observation reflects a general feature of the MUBs, it may contribute to the understanding of why no larger sets of MUBs are found in six dimensions.

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